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On the Linear Damped Wave Equation

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In this work we estimate the spectrum of the linear damped wave semigroup under homogeneous Dirichlet boundary conditions by using the principal eigenvalue of an elliptic operator related to the equation. Our estimate is optimal for real eigenvalues. Then, we analyze the behavior of the estimate as the *damping amplitude* grows to infinity. When the damping changes of sign we extend a result of Freitas [5] to show that the semigroup possesses at least two real eigenvalues greater than

This analysis is based upon the behavior of the principal eigenpair of a singular perturbation problem at the singular limit. Our theory is of interest by itself and it has many applications to reaction diffusion systems (c.f. for instance [6]).

1. INTRODUCTION

This paper is motivated by the following evolution problem

$$
\begin{aligned}
u_{tt}+D(x)\,u_t+\mathscr{L}u&=0, && (x,t)\in\Omega\times(0,\infty)\\
u\,|_{\partial\Omega}&=0, && t>0\\
(u(\cdot,0),v(\cdot,0))&=(u_0,v_0), && v=u_t
\end{aligned}
\tag{1.1}
$$

for the linear damped wave equation, where $\Omega\subset\mathbb{R}^N$, $N\geqslant 1$, is a bounded domain whose boundary $\partial\Omega$ satisfies the uniform interior ball condition, $\mathscr{L}$ is a strongly uniformly elliptic differential operator in $\bar{\Omega}$ of the form

$$\mathscr{L}=-\sum_{i,j=1}^{N} D_j(\alpha_{ij}(x)\,D_i)+\sum_{j=1}^{N}\alpha_j(x)\,D_j+\alpha_0(x), \tag{1.2}$$

with coefficients α_{ij}, $\alpha_j\in C^1(\bar{\Omega})$, $\alpha_{ij}=\alpha_{ji}$, $i,j\in\{1,...,N\}$, $\alpha_0\in C(\bar{\Omega})$, and $D\in C(\bar{\Omega})$ is a sign indefinited damping. By strongly uniformly elliptic we mean that there exists a positive constant $\nu>0$ such that

$$\sum_{i,j=1}^{N}\alpha_{ij}(x)\,\xi_i\xi_j\geqslant\nu\,|\xi|^2 \tag{1.3}$$

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for all $x \in \bar{\Omega}$ and $\xi = (\xi_1, ..., \xi_N) \in \mathbb{R}^N$. Under these assumptions (1.1) is well-posed in the space $H_0^1(\Omega) \times L^2(\Omega)$. In addition, throughout this work we shall assume that $\sigma_1^\Omega[\mathscr{L}] > 0$, where $\sigma_1^\Omega[\mathscr{L}]$ stands for the principal eigenvalue of $\mathscr{L}$ in Ω under homogeneous Dirichlet boundary conditions. It is known that $\sigma_1^\Omega[\mathscr{L}] > 0$ if, and only if, $\mathscr{L}$ satisfies the strong maximum principle, [11], [12]. In the sequel we shall write $D(x)$ in the form

$$D(x) = \varepsilon A(x), \qquad \varepsilon \in \mathbb{R}^+, \qquad \max_{x \in \bar{\Omega}} A = 1, \tag{1.4}$$

and will regard to the *damping amplitude*, ε, as a parameter.

The stability of the zero solution of (1.1) is given by the real parts of the eigenvalues of the operator

$$\mathscr{A} := \begin{pmatrix} 0 & I \\ -\mathscr{L} & -D(x) \end{pmatrix}, \qquad D(\mathscr{A}) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega). \tag{1.5}$$

Note that $z \in \mathbb{C}$ is an eigenvalue of $\mathscr{A}$ with associated eigenfunction (U, V) if, and only if, $V = zU$ and zero is an eigenvalue of the elliptic operator

$$\mathscr{L} + D(x)\, z + z^2,$$

with associated eigenfunction V. If $D > 0$, then the results of Dafermos [3] and Haraux [7] can be adapted to show that the energy of every solution of (1.1) decays to zero as $t \to \infty$. Moreover, in some special cases it is known that the energy decays exponentially (c.f. Section 5 of Bardos *et al.* [1] and Zuazua [13]). The fact that in the one-dimensional model the optimal decay exponent equals to the type of the semigroup generated by $\mathscr{A}$ was shown by Cox and Zuazua, [2]. In several space dimensions the type of the semigroup does not coincide in general with the optimal decay rate as it was shown by Lebeau [10]. If $D(x)$ changes sign, then the situation changes drastically. Indeed, it was shown by Freitas in [5] that if $\mathscr{L} = -\Delta$ and ε is sufficiently large, then the operator $\mathscr{A}$ possesses at least two positive real eigenvalues; in particular the trivial solution of (1.1) is unstable. The problem of characterizing whether stabilization to zero occurs remains open.

Although all the results of the paper will be obtained for (1.2), to motivate our analysis we shall assume that $\mathscr{L}$ is self-adjoint, i.e. $\alpha_j = 0$ for all j. Then, we have the following variational characterization

$$\sigma_1^\Omega[\mathscr{L} + \lambda D] = \inf_{\psi \in H_0^1(\Omega),\, \int_\Omega \psi^2 = 1} \left\{ \sum_{i,j=1}^N \int_\Omega \alpha_{ij}\, D_i\psi\, D_j\psi + \int_\Omega (\alpha_0 + \lambda D)\, \psi^2 \right\}, \tag{1.6}$$

for any $\lambda \in \mathbb{R}$. Suppose that $\lambda + i\mu \in \mathbb{C}$ is an eigenvalue of $\mathscr{A}$ and let $\varphi + i\psi$ be an eigenfunction associated with the zero eigenvalue of

$$\mathscr{L} + D(\lambda + i\mu) + (\lambda + i\mu)^2.$$

Here i stands for the complex imaginary unit. Then,

$$\begin{aligned}(\mathscr{L} + \lambda D + \lambda^2 - \mu^2)\,\varphi &= (\mu D + 2\lambda\mu)\,\psi,\\ (\mathscr{L} + \lambda D + \lambda^2 - \mu^2)\,\psi &= -(\mu D + 2\lambda\mu)\,\varphi.\end{aligned} \tag{1.7}$$

Multiplying the first equation of (1.7) by φ, the second by ψ, adding the resulting relations, integrating over Ω, and applying the formula of integration by parts we find that

$$\begin{aligned}&\sum_{i,j=1}^{N} \int_\Omega \alpha_{ij}\, D_i\varphi\, D_j\varphi + \int_\Omega (\lambda D + \alpha_0)\,\varphi^2\\ &\quad + \sum_{i,j=1}^{N} \int_\Omega \alpha_{ij}\, D_i\psi\, D_j\psi + \int_\Omega (\lambda D + \alpha_0)\,\psi^2 = (\mu^2 - \lambda^2) \int_\Omega (\varphi^2 + \psi^2).\end{aligned}$$

Thus, it follows from (1.6) that

$$(\mu^2 - \lambda^2) \int_\Omega (\varphi^2 + \psi^2) \geqslant \sigma_1^\Omega[\mathscr{L} + \lambda D] \int_\Omega (\varphi^2 + \psi^2),$$

and therefore,

$$\mu^2 \geqslant \max\{\sigma_1^\Omega[\mathscr{L} + \lambda D + \lambda^2], 0\}. \tag{1.8}$$

This shows that the graph of the function $\mathscr{P}(\lambda)$ defined by

$$\mathscr{P}(\lambda) := \sigma_1^\Omega[\mathscr{L} + \lambda D + \lambda^2] \tag{1.9}$$

provides us with an estimate of the region where the spectrum of $\mathscr{A}$ is located. The estimate (1.8) is optimal in the sense that the equality holds at any real eigenvalue of $\mathscr{A}$. So, the interest of the problem of analyzing the graph of $\mathscr{P}(\lambda)$. If the damping is a positive constant, then this analysis is straightforward and the existence of a real eigenvalue of $\mathscr{L}$ can be characterized by means of $\sigma_1^\Omega[\mathscr{L}]$. Further, using Faber–Krahn inequality, $\sigma_1^\Omega[\mathscr{L}]$ can be estimated in terms of $|\Omega|$, where $|\cdot|$ stands for the Lebesgue measure. We shall do this analysis in Sections 2, 3. Section 2 is devoted to the case $A \equiv 1$ and Section 3 to the general case $0 < A \leqslant 1$. In these sections we shall give some very general estimates for the negative real eigenvalues of $\mathscr{A}$. Our estimates are essentially new and provide us with substantial extensions of those found by Cox and Zuazua [2] for intervals and two-dimensional simply connected domains.

In Section 5 we treat the case of sign indefinited dampings when $\mathscr{L}$ admits a transformation into a selfadjoint operator in the sense of Theorem 4.1. First, we shall extend the main result of Freitas [5] showing that if ε is sufficiently large, then $\mathscr{A}$ possesses at least four real eigenvalues, two positive and two negative. Then, we shall analyze the behavior of the real eigenvalues and their associated eigenfunctions as $\varepsilon \to \infty$. Even in the simplest situations, these asymptotic results seems to be unknown. For instance, in Remark 2.1 of [5] it was claimed that

$$\lim_{p \to \infty} \sigma_1^{\Omega}[-\Delta + pa + b] = \infty,$$

when "$a(x)$ is of definite sign". This is true if we assume that $a(x) > 0$ almost everywhere in Ω, but fails to be true when $a \equiv 0$ in some subdomain of Ω. In this case the previous limit is finite and it is given by the principal eigenvalue of $-\Delta + b$ in the region where a vanishes, [11], [12].

We now shortly describe some of the results that we have found in Section 5. Assume that $A(x)$ changes of sign and let $\lambda_{1,+}(\varepsilon) < \lambda_{2,+}(\varepsilon)$ denote the two positive zeros of $\mathscr{P}(\lambda)$ such that $\mathscr{P}(\lambda) \geqslant 0$ if $\lambda \in [0, \lambda_{1,+}(\varepsilon))$ or $\lambda \in (\lambda_{2,+}(\varepsilon), \infty)$. Theorem 5.1 shows that these zeros are well defined if ε is sufficiently large. It can be easily seen that the following holds

$$\lim_{\varepsilon \uparrow \infty} \lambda_{1,+}(\varepsilon) = 0, \qquad \lim_{\varepsilon \uparrow \infty} \lambda_{2,+}(\varepsilon) = \infty.$$

In fact, this is contained in Proposition 3.8 of Freitas [5] if $\mathscr{L} = -\Delta$. In Section 5 we sall show that

$$\lim_{\varepsilon \uparrow \infty} \varepsilon \lambda_{1,+}(\varepsilon) = \sigma_{1,+}^{\Omega}[\mathscr{L}; -A], \qquad \lim_{\varepsilon \uparrow \infty} \frac{\lambda_{2,+}(\varepsilon)}{\varepsilon} = -\inf_{\Omega} A,$$

where $\sigma_{1,+}^{\Omega}[\mathscr{L}; -A]$ stands for the positive principal eigenvalue of the operator $\mathscr{L}$ with respect to the weight $-A$. By a principal eigenvalue we mean an eigenvalue associated with it there is a positive eigenfunction (see Remark 5.2 in Section 5). Moreover, if we denote by $\varphi_{j,+,\varepsilon}$ the principal eigenfunctions associated with $\lambda_{j,+}(\varepsilon)$, i.e.

$$[\mathscr{L} + \varepsilon A(x)\, \lambda_{j,+}(\varepsilon) + \lambda_{j,+}^2(\varepsilon)]\, \varphi_{j,+,\varepsilon} = 0, \qquad j = 1, 2,$$

normalized so that

$$\int_{\Omega} |\nabla \varphi_{j,+,\varepsilon}|^2 = 1,$$

then, for any sequence $\varepsilon_n \to \infty$, $n \to \infty$, there exist two subsequences (again labeled by n) such that

$$\lim_{n \to \infty} \varphi_{j,\,+,\,\varepsilon_n} = \varphi_{j,\,+,\,\infty}, \qquad j = 1, 2,$$

in $L^2(\Omega)$, for some $\varphi_{j,\,+,\,\infty} \in L^2(\Omega)$. Furthermore, the following holds:

(i) $\varphi_{2,\,+,\,\infty} \equiv 0$ in the region where $A > \inf_\Omega A$,

(ii) $\varphi_{1,\,+,\,\infty}$ is the principal eigenfunction associated with $\sigma^\Omega_{1,\,+}[\mathscr{L}; -A]$.

(iii) $\lim_{\varepsilon \uparrow \infty} \varphi_{1,\,+,\,\varepsilon} = \varphi_{1,\,+,\,\infty}$ in $L^2(\Omega)$.

Since A changes of sign, the same result holds if we consider, instead of positive, negative eigenvalues of $\mathscr{A}$. Therefore, we have characterized the asymptotic behavior of these eigenvalues and eigenfunctions.

To make the previous analysis we study how varies the principal eigenpair of an operator of the form $d\mathscr{L} + V(x)$ as $d \downarrow 0$. This analysis will be done in Section 4 (see Theorem 4.1). In [6] we found the behavior of $\sigma_1[-d\Delta + V(x)]$ as $d \downarrow 0$. Here, we extend this result to cover a general class of differential operators and in addition we find how behaves the principal eigenfunction as $d \downarrow 0$. This analysis is new and of great interest by itself. So, we have included it into a separate section.

2. THE CASE OF A CONSTANT DAMPING

Throughout this section we assume that $A \equiv 1$. Then, we have that the function $\mathscr{P}(\lambda)$ defined by (1.9) is given by

$$\mathscr{P}(\lambda) = \sigma^\Omega_1[\mathscr{L}] + \varepsilon\lambda + \lambda^2.$$

Therefore, it is a parabola and the following result holds.

COROLLARY 2.1. *There exists $\lambda_1 \in \mathbb{R}$ such that $\mathscr{P}(\lambda_1) = 0$ if, and only if,*

$$\varepsilon^2 \geqslant 4\sigma^\Omega_1[\mathscr{L}]. \tag{2.1}$$

Moreover, if $\varepsilon^2 = 4\sigma^\Omega_1[\mathscr{L}]$, then $\lambda_1 = -\varepsilon/2$ is the unique zero of $\mathscr{P}(\lambda)$, and if $\varepsilon^2 > 4\sigma^\Omega_1[\mathscr{L}]$, then

$$\lambda^\pm_1 = \tfrac{1}{2}\left(-\varepsilon \pm \sqrt{\varepsilon^2 - 4\sigma^\Omega_1[\mathscr{L}]}\right) < 0,$$

are the two unique zeros of $\mathscr{P}(\lambda)$.

Observe that many other real eigenvalues may appear when ε is large enough. In fact, if σ_j is a real eigenvalue of $\mathcal{L}$ and $\varepsilon^2 > 4\sigma_j$, then $\mathcal{P}(\lambda_j^{\pm}) = 0$ where

$$\lambda_j^{\pm} = \tfrac{1}{2}(-\varepsilon \pm \sqrt{\varepsilon^2 - 4\sigma_j}) < 0,$$

If the Lebesgue measure of the domain, $|\Omega|$, is sufficiently small, then $\sigma_1^{\Omega}[\mathcal{L}]$ is arbitrarily large and hence condition (2.1) will never be satisfied. The following result makes precise how small has to be $|\Omega|$ so that

$$\varepsilon^2 < 4\sigma_1^{\Omega}[\mathcal{L}]. \tag{2.2}$$

THEOREM 2.2. (i) *If $\varepsilon^2 \leqslant 4 \inf_{\Omega} \alpha_0$, then the condition*

$$|\Omega| < \left(\frac{\nu \sqrt{\sigma_1^{B_1}[-\Delta]}}{|\vec{\alpha}|_{2,\infty}} \right)^N |B_1| \tag{2.3}$$

guarantees that $\mathcal{P}(\lambda) > 0$ for all $\lambda \in \mathbb{R}$. We have denoted by B_1 to the unit ball of $\mathbb{R}^N$, $\vec{\alpha} = (\alpha_1, ..., \alpha_N)$ and

$$|\vec{\alpha}|_{2,\infty} := \sup_{x \in \Omega} \left(\sum_{j=1}^{N} \alpha_j^2(x) \right)^{1/2}.$$

(ii) *If $\varepsilon^2 > 4 \inf_{\Omega} \alpha_0$, then the condition*

$$|\Omega| < \left(\frac{2\nu \sqrt{\sigma_1^{B_1}[-\Delta]}}{|\vec{\alpha}|_{2,\infty} + \sqrt{|\vec{\alpha}|_{2,\infty}^2 - \nu(4 \inf_{\Omega} \alpha_0 - \varepsilon^2)}} \right)^N |B_1| \tag{2.4}$$

guarantees that $\mathcal{P}(\lambda) > 0$ for all $\lambda \in \mathbb{R}$.

If $\mathcal{L}$ is selfadjoint, i.e., if $\vec{\alpha} \equiv 0$, then (2.3) *should be readed as $|\Omega| < \infty$, which is always satisfied.*

Proof. We shall show that (2.3) and (2.4) imply (2.2). Let $\varphi \gg 0$ denote the principal eigenfunction corresponding to $\sigma_1^{\Omega}[\mathcal{L}]$. Multiplying the equation $\mathcal{L}\varphi = \sigma_1^{\Omega}[\mathcal{L}]\varphi$ by φ integrating in Ω and applying the formula of integration by parts, we find that

$$\sigma_1^{\Omega}[\mathcal{L}] \int_{\Omega} \varphi^2 = \sum_{i,j=1}^{N} \int_{\Omega} \alpha_{ij} D_i\varphi\, D_j\varphi + \sum_{j=1}^{N} \int_{\Omega} \alpha_j \varphi\, D_j\varphi + \int_{\Omega} \alpha_0 \varphi^2. \tag{2.5}$$

It follows from (1.3) that

$$\sum_{i,j=1}^{N} \int_{\Omega} \alpha_{ij} D_i\varphi\, D_j\varphi \geqslant \nu \int_{\Omega} |\nabla\varphi|^2. \tag{2.6}$$

Moreover, Hölder's inequality implies that

$$\left|\sum_{j=1}^{N}\int_\Omega \alpha_j\varphi\, D_j\varphi\right| = \left|\int_\Omega \varphi\,\langle\vec{\alpha},\nabla\varphi\rangle\right| \leqslant \int_\Omega |\varphi|\cdot|\langle\vec{\alpha},\nabla\varphi\rangle|$$

$$\leqslant |\vec{\alpha}|_{2,\infty}\int_\Omega |\varphi|\cdot|\nabla\varphi|_2 \leqslant |\vec{\alpha}|_{2,\infty}\,\|\varphi\|_2\,\|\nabla\varphi\|_2.$$

Hence,

$$\sum_{j=1}^{N}\int_\Omega \alpha_j\varphi\, D_j\varphi \geqslant -|\vec{\alpha}|_{2,\infty}\,\|\varphi\|_2\,\|\nabla\varphi\|_2. \tag{2.7}$$

Substituting (2.6) and (2.7) into (2.5) yields

$$\sigma_1^\Omega[\mathscr{L}] \geqslant \left(\frac{\int_\Omega |\nabla\varphi|^2}{\int_\Omega \varphi^2}\right)^{1/2}\left[\nu\left(\frac{\int_\Omega |\nabla\varphi|^2}{\int_\Omega \varphi^2}\right)^{1/2} - |\vec{\alpha}|_{2,\infty}\right] + \inf_\Omega \alpha_0.$$

Setting

$$X := \left(\frac{\int_\Omega |\nabla\varphi|^2}{\int_\Omega \varphi^2}\right)^{1/2},$$

the previous relation can be written in the form

$$\sigma_1^\Omega[\mathscr{L}] \geqslant X(\nu X - |\vec{\alpha}|_{2,\infty}) + \inf_\Omega \alpha_0.$$

Thus, it suffices to show that under conditions (2.3) or (2.4) we have

$$X(\nu X - |\vec{\alpha}|_{2,\infty}) + \inf_\Omega \alpha_0 - \frac{\varepsilon^2}{4} > 0. \tag{2.8}$$

From the variational characterization of $\sigma_1^\Omega[-\Delta]$ we find that $X^2 \geqslant \sigma_1^\Omega[-\Delta]$. Moreover, due to the Faber–Krahn inequality $\sigma_1^\Omega[-\Delta] \geqslant \sigma_1^{B_R}[-\Delta]$, where B_R is the ball centered at the origin with radio R so that $|B_R| = |\Omega|$, [4], [9]. We have $\sigma_1^{B_R}[-\Delta] = (1/R^2)\,\sigma_1^{B_1}[-\Delta]$ and $|B_R| = |B_1|\,R^N$. Thus,

$$X \geqslant \sqrt{\sigma_1{}^{B_1}[-\Delta]}\left(\frac{|B_1|}{|\Omega|}\right)^{1/N}. \tag{2.9}$$

Suppose that $\varepsilon^2 \leqslant 4\inf_\Omega \alpha_0$. Then, (2.8) holds as soon as $X > \nu^{-1}|\vec{\alpha}|_{2,\infty}$. Thus, thanks to (2.9), if we assume that

$$\sqrt{\sigma_1^{B_1}[-\Delta]}\left(\frac{|B_1|}{|\Omega|}\right)^{1/N} > \nu^{-1}\,|\vec{\alpha}|_{2,\infty}, \tag{2.10}$$

then (2.8), and hence (2.2), holds. As (2.10) is the same as (2.3), the proof of Part (i) is completed.

Now, assume that $\varepsilon^2 > 4 \inf_\Omega \alpha_0$. Then, due to (2.9), it follows from (2.4) that

$$X > \frac{1}{2\nu} \left[|\vec{\alpha}|_{2,\infty} + \sqrt{|\vec{\alpha}|_{2,\infty}^2 - \nu(4 \inf_\Omega \alpha_0 - \varepsilon^2)} \right],$$

which is the largest root of the left hand side of (2.8). Therefore, (2.8) holds. This completes the proof. ∎

It is straightforward to see that the converse of Theorem 2.2 is also true in dimension $N=1$, in the sense that if the measure of Ω is large enough, then $\mathscr{P}(\lambda)$ possesses a real zero. The following example shows that this is not necessarily the case if $N \geqslant 2$. For these dimensions the fact that condition (2.1) holds, or not, is intimately related to the shape of the domain and not only to its measure.

Let L_j, $1 \leqslant j \leqslant N$, be $N \geqslant 2$ positive constants. Let $\Omega := \times_{j=1}^N (0, L_j)$. Then,

$$|\Omega| = \Pi_{j=1}^N L_j, \qquad \sigma_1^\Omega[-\Delta] = \sum_{j=1}^N \frac{\pi^2}{L_j^2}.$$

Suppose that

$$\varepsilon^2 < 4 \sum_{j=1}^{N-1} \frac{\pi^2}{L_j^2}.$$

Then, for any $L_N > 0$ we have that $\varepsilon^2 < 4\sigma_1^\Omega[-\Delta]$ and therefore $\mathscr{P}(\lambda) > 0$ for all $\lambda \in \mathbb{R}$; thought $|\Omega|$ may assume any positive value as L_N varies from 0 to ∞. This phenomenology is inherent to the shape of Ω. Indeed, if we take $\Omega := B_R$, then

$$|\Omega| = |B_1|\, R^N, \qquad \sigma_1^\Omega[-\Delta] = R^{-2} \sigma_1^{B_1}[-\Delta],$$

and hence, $|\Omega| \to \infty$ if, and only if, $R \to \infty$, i.e. if, and only if, $\sigma_1^\Omega[-\Delta] \to 0$. Thus, if the measure of $|\Omega|$ is large enough, then condition (2.1) holds and therefore $\mathscr{P}(\lambda)$ changes of sign in $\mathbb{R}$.

Such examples show that the estimate of $\sigma_1^\Omega[-\Delta]$ by means of $|\Omega|$ given by Faber–Krahn inequality may be very bad if the domain is far away from a ball.

3. THE CASE OF A VARIABLE DAMPING WITH DEFINITE SIGN

In this section we assume that $A > 0$ is arbitrary. Remember that we have normalized it so that $\sup_\Omega A = 1$. The following result holds.

THEOREM 3.1. (i) *If* (2.2) *is satisfied, then* $\mathscr{P}(\lambda) > 0$ *for all* $\lambda \in \mathbb{R}$.

(ii) *Given* $\delta > 0$, *let* S_δ *denote the set of subdomains* $\Omega_\delta \subset \Omega$ *such that* $A \geqslant \delta$ *in* Ω_δ. *Suppose that*

$$\varepsilon^2 > 4 \inf_{\delta > 0,\, \Omega_\delta \in S_\delta} \frac{\sigma_1^{\Omega_\delta}[\mathscr{L}]}{\delta^2}. \tag{3.1}$$

Then, $\mathscr{P}(\lambda)$ *changes of sign.*

Proof. Since $A > 0$ and $\sigma_1^{\Omega}[\mathscr{L}] > 0$ we have that $\mathscr{P}(\lambda) > 0$ for all $\lambda \geqslant 0$. Moreover, since $A \leqslant 1$, we find that for any $\lambda < 0$

$$\sigma_1^{\Omega}[\mathscr{L} + \varepsilon A \lambda + \lambda^2] \geqslant \sigma_1^{\Omega}[\mathscr{L} + \varepsilon \lambda + \lambda^2].$$

If (2.2) holds, then the right hand side of this inequality is positive. This completes the proof of Part (i). If we assume (3.1), then $\varepsilon^2 \delta^2 > 4\sigma_1^{\Omega_\delta}[\mathscr{L}]$ for some $\Omega_\delta \subset \Omega$ such that $A \geqslant \delta > 0$ in Ω_δ. On the other hand, by the monotonicity of the principal eigenvalue with respect to the domain, we find that

$$\sigma_1^{\Omega}[\mathscr{L} + \varepsilon A \lambda + \lambda^2] \leqslant \sigma_1^{\Omega_\delta}[\mathscr{L} + \varepsilon A \lambda + \lambda^2]$$

and hence

$$\sigma_1^{\Omega}[\mathscr{L} + \varepsilon A \lambda + \lambda^2] \leqslant \sigma_1^{\Omega_\delta}[\mathscr{L}] + \varepsilon \delta \lambda + \lambda^2$$

for all $\lambda < 0$. As $\varepsilon^2 \delta^2 > 4\sigma_1^{\Omega_\delta}[\mathscr{L}]$, the right hand side of the last inequality possesses a real root. This completes the proof. ∎

We now estimate the zeros of $\mathscr{P}(\lambda)$ by using the same technique as in the proof of Theorem 2.2.

THEOREM 3.2. *Suppose that*

$$|\vec{\alpha}|_{2,\infty} < v \sqrt{\sigma_1^{B_1}[-\Delta]} \left(\frac{|B_1|}{|\Omega|} \right)^{1/N}. \tag{3.2}$$

Let λ_1 *be such that* $\mathscr{P}(\lambda_1) = 0$. *Then,*

$$\lambda_1 \in [\lambda_1^-, \lambda_1^+],$$

where

$$\lambda_1^{\pm} := \frac{\varepsilon}{2} \pm \left[\frac{\varepsilon^2}{4} + |\vec{\alpha}|_{2,\infty} \sqrt{\sigma_1^{B_1}[-\Delta]} \left(\frac{|B_1|}{|\Omega|} \right)^{1/N} - v\sigma_1^{B_1}[-\Delta] \left(\frac{|B_1|}{|\Omega|} \right)^{2/N} - \inf_{\Omega} \alpha_0 \right]^{1/2}.$$

Proof. Since $A > 0$, necessarily $\lambda_1 < 0$. Thus, we can restrict ourselves to consider $\lambda < 0$. For such λ's, since $A \leqslant 1$, we find that

$$\mathscr{P}(\lambda) = \sigma_1^{\Omega}[\mathscr{L} + \varepsilon A\lambda + \lambda^2] \geqslant \sigma_1^{\Omega}[\mathscr{L}] + \varepsilon\lambda + \lambda^2.$$

Thus, the same argument as in the proof of Theorem 2.2, using in addition (3.2), shows that

$$\mathscr{P}(\lambda) \geqslant \lambda^2 + \varepsilon\lambda + \inf_{\Omega} \alpha_0 + v\sigma_1^{B_1}[-\Delta] \left(\frac{|B_1|}{|\Omega|}\right)^{2/N}$$
$$- |\vec{\alpha}|_{2,\infty} \sqrt{\sigma_1^{B_1}[-\Delta]} \left(\frac{|B_1|}{|\Omega|}\right)^{1/N}. \tag{3.3}$$

Therefore, the zeros of $\mathscr{P}(\lambda)$ must lye in between the zeros of the right hand side of (3.3), that are $\lambda_1^{\pm}$. This completes the proof. ∎

If $\mathscr{L} = -\Delta$, then (3.3) becomes into

$$\mathscr{P}(\lambda) \geqslant \lambda^2 + \varepsilon\lambda + \sigma_1^{B_1}[-\Delta] \left(\frac{|B_1|}{|\Omega|}\right)^{2/N}.$$

By brute force, this estimate yields

$$\mathscr{P}(\lambda) \geqslant \varepsilon\lambda + \sigma_1^{B_1}[-\Delta] \left(\frac{|B_1|}{|\Omega|}\right)^{2/N}$$

and therefore,

$$\lambda \leqslant -\frac{\sigma_1^{B_1}[-\Delta]}{\varepsilon} \left(\frac{|B_1|}{|\Omega|}\right)^{2/N}. \tag{3.4}$$

Note that if $A \equiv 1$ and $N = 2$, then (3.4) equals the estimate given in Theorem 4.7 of [2], which was found for simply connected domains. Therefore, our estimate is substantially sharper.

4. LIMITING BEHAVIOR OF PRINCIPAL EIGENVALUES AND EIGENFUNCTIONS

In [6] we found the limiting behavior of $\sigma_1{}^{\Omega}[-d\Delta+p]$ as $d\downarrow 0$. Here we extend the result of [6]to a general class of operators analyzing in addition how varies the principal eigenfunction as $d\downarrow 0$. The following result holds.

THEOREM 4.1. *Suppose that there exists $\beta \in C^2(\Omega)$ such that*

$$\alpha_j = 2\sum_{i=1}^{N} \alpha_{ij} D_i \beta, \qquad 1 \leqslant j \leqslant N. \tag{4.1}$$

Let $p \in C(\bar{\Omega})$ be. Then

$$\lim_{d\downarrow 0} \sigma_1^{\Omega}[d\mathscr{L}+p] = \inf_{\Omega} p. \tag{4.2}$$

Let φ_d denote the principal eigenfunction associated with $\sigma_1^{\Omega}[d\mathscr{L}+p]$, normalized so that

$$\int_{\Omega} |\nabla \varphi_d|^2 = 1. \tag{4.3}$$

Then, for any sequence $d_n \downarrow 0$, as $n\uparrow\infty$, there exists a subsequence (related by n) such that

$$\lim_{n\to\infty} \varphi_{d_n} = \varphi_{\infty}, \qquad in\ L^2(\Omega),$$

for some $\varphi_{\infty} \in L^2(\Omega)$ such that

$$\varphi_{\infty} \equiv 0 \qquad in \quad p > \inf_{\Omega} p. \tag{4.4}$$

Suppose that, for instance, p reaches its minimum on a finite set. Then, it is clear that $\varphi_{\infty}=0$. Moreover, it follows from (4.2) that

$$\lim_{n\to\infty} \|\varphi_{d_n}\|_{H^1(\Omega)} = 1.$$

Thus, (φ_d) does not admit a convergent subsequence in $H^1(\Omega)$ and therefore $(\nabla\varphi_{d_n})$ does not admit a convergent subsequence in $L^2(\Omega)$. We conjecture that if p reaches its minimum on a single point x_0, then

$$\lim_{n\to\infty} \nabla\varphi_{d_n} = \delta_{x_0},$$

in the sense of distributions. If the minimum is not reached on a single point, then we can not be that sure about the uniqueness of this limit. We emphasize that this lack of convergence in $H^1(\Omega)$ is the reason why the analysis in this section is interesting from the mathematical point of view, beside eventual physical applications. We are analyzing the behavior of the principal eigenpair when passing to the limit in a singular perturbation problem.

Remark 4.2. Condition (4.1) holds if for instance $\mathscr{L}$ is self-adjoint, i.e. if $\alpha_j = 0$ for all j. In this case it suffices to take $\beta = 0$. Condition (4.1) is also satisfied when the principal part is the Laplacian and α is a curl-free vector field.

Proof of Theorem 4.1. First we shall prove the theorem when $\mathscr{L}$ is selfadjoint. Then, we will prove the general case. Suppose that $\vec{\alpha} = 0$. Then,

$$\sigma_1^\Omega[d\mathscr{L} + p] = \inf_{\psi \in H_0^1(\Omega),\, \int_\Omega \psi^2 = 1} \left\{ d \sum_{i,j=1}^{N} \int_\Omega \alpha_{ij}\, D_i\psi\, D_j\psi + \int_\Omega (d\alpha_0 + p)\, \psi^2 \right\}. \tag{4.5}$$

Moreover, the infimum is reached at the principal eigenfunction φ_d. Let $\psi \in H_0^1(\Omega)$ be such that $\int_\Omega \psi^2 = 1$. Then, using the variational characterization of $\sigma_1^\Omega[-\Delta]$ it follows from (2.6) that

$$\begin{aligned} d \sum_{i,j=1}^{N} \int_\Omega \alpha_{ij}\, D_i\psi\, D_j\psi + \int_\Omega (d\alpha_0 + p)\psi^2 &\geqslant dv \int_\Omega |\nabla\psi|^2 + \inf_\Omega (d\alpha_0 + p) \\ &\geqslant dv\, \sigma_1^\Omega[-\Delta] + d \inf_\Omega \alpha_0 + \inf_\Omega p. \end{aligned}$$

Thus, taking infimums in the left hand side of this inequality, we find from (4.5) that

$$\sigma_1^\Omega[d\mathscr{L} + p] \geqslant dv\, \sigma_1^\Omega[-\Delta] + d \inf_\Omega \alpha_0 + \inf_\Omega p.$$

Therefore,

$$\liminf_{d \downarrow 0} \sigma_1^\Omega[d\mathscr{L} + p] \geqslant \inf_\Omega p.$$

Now, pick $\varepsilon > 0$ arbitrary and $\psi \in H_0^1(\Omega)$ such that $\int_\Omega \psi^2 = 1$ and $\psi \equiv 0$ in the region where $p \geqslant \inf_\Omega p + \varepsilon$, i.e. with its mass concentrated in the region where $\inf_\Omega p \leqslant p(x) \leqslant \inf_\Omega p + \varepsilon$. Then, it follows from (4.5) that

$$\sigma_1^\Omega[d\mathscr{L} + p] \leqslant d \sum_{i,j=1}^{N} \int_\Omega \alpha_{ij}\, D_i\psi\, D_j\psi + d \sup_\Omega \alpha_0 + \inf_\Omega p + \varepsilon.$$

Hence,

$$\limsup_{d \downarrow 0} \sigma_1^\Omega[d\mathscr{L} + p] \leqslant \inf_\Omega p + \varepsilon.$$

This shows (4.2) when $\mathscr{L}$ is selfadjoint. Notice that this argument also shows that if p_d, $d>0$, is a family of functions in $C(\bar{\Omega})$ such that $\lim_{d\downarrow 0} p_d=0$ in $L^\infty(\Omega)$, then

$$\lim_{d\downarrow 0} \sigma_1^\Omega[d\mathscr{L}+p+p_d]=\inf_\Omega p. \tag{4.6}$$

We now show (4.2) for a general operator $\mathscr{L}$, not necessarily selfadjoint. Let $\beta\in C^2(\Omega)$ be satisfying (4.1). Then, the change of variable

$$\varphi_d=e^{\beta(x)}\psi_d \tag{4.7}$$

transforms

$$d\mathscr{L}\varphi_d+p\varphi_d=\sigma_1^\Omega[d\mathscr{L}+p]\,\varphi_d$$

into

$$-d\sum_{i,j=1}^N D_j(\alpha_{ij}\,D_i\psi_d)+dq_\beta\psi_d+p\psi_d=\sigma_1^\Omega[d\mathscr{L}+p]\,\psi_d, \tag{4.8}$$

where

$$q_\beta:=\alpha_0+\sum_{j=1}^N \alpha_j\,D_j\beta-\sum_{i,j=1}^N\,[D_j(\alpha_{ij}\,D_i\beta)+\alpha_{ij}\,D_j\beta\,D_i\beta]. \tag{4.9}$$

Since the differential operator in the left hand side of (4.8) is selfadjoint and q_β does not depend on d we have that

$$\lim_{d\downarrow 0}\sigma_1^\Omega\left[-d\sum_{i,j=1}^N D_j(\alpha_{ij}\,D_i\,\cdot\,)+dq_\beta+p\right]=\inf_\Omega p.$$

On the other hand, since the operator in the left hand side of (4.8) possesses a unique eigenvalue to a positive eigenfunction and $\psi_d>0$, we find that

$$\sigma_1^\Omega\left[-d\sum_{i,j=1}^N D_j(\alpha_{ij}\,D_i\,\cdot\,)+dq_\beta+p\right]=\sigma_1^\Omega[d\mathscr{L}+p].$$

This completes the proof of (4.2).

To complete the proof of the theorem it suffices to show that ψ_d satisfies the same properties of the statement of the theorem as φ_d. Multiplying (4.8) by ψ_d and integrating by parts we find that

$$d\sum_{i,j=1}^N\int_\Omega \alpha_{ij}\,D_i\psi_d\,D_j\psi_d+d\int_\Omega q_\beta\psi_d^2+\int_\Omega p\psi_d^2=\sigma_1^\Omega[d\mathscr{L}+p]\int_\Omega \psi_d^2. \tag{4.10}$$

Since φ_d satisfies (4.3), it follows from the variational characterization of $-\Delta$ that

$$\sigma_1^{\Omega}[-\Delta]\int_{\Omega}\varphi_d^2 \leqslant 1.$$

Hence, $(\varphi_d)_{d>0}$ is bounded in $H_0^1(\Omega)$ and therefore from any sequence (d_n), $n\geqslant 1$, $d_n \downarrow 0$, we can extract a subsequence, relabeled by n, such that

$$\lim_{n\to\infty}\|\varphi_{d_n}-\varphi_{\infty}\|_{L^2(\Omega)}=0,$$

for some $\varphi_{\infty}\in L^2(\Omega)$. It follows from (4.7) that

$$\lim_{n\to\infty}\|\psi_{d_n}-\psi_{\infty}\|_{L^2(\Omega)}=0,$$

where

$$\psi_{\infty}=e^{-\beta}\varphi_{\infty}. \tag{4.11}$$

Using (2.6), we find from (4.10) that

$$\begin{aligned}\sigma_1^{\Omega}[d_n\mathscr{L}+p]\int_{\Omega}\psi_{d_n}^2 &\geqslant d_n\nu\int_{\Omega}|\nabla\psi_{d_n}|^2+d_n\int_{\Omega}q_{\beta}\psi_{d_n}^2+\int_{\Omega}p\psi_{d_n}^2\\ &\geqslant d_n\nu\sigma_1^{\Omega}[-\Delta]\int_{\Omega}\psi_{d_n}^2+d_n\int_{\Omega}q_{\beta}\psi_{d_n}^2+\int_{\Omega}p\psi_{d_n}^2.\end{aligned}$$

Now, passing to the limit as $n\to\infty$ in this relation, we obtain that

$$\inf_{\Omega}p\int_{\Omega}\psi_{\infty}^2\geqslant\int_{\Omega}p\psi_{\infty}^2.$$

Thus, $\psi_{\infty}\equiv 0$ in the region $p>\inf_{\Omega}p$. Finally, we see from (4.11) that the same holds for φ_{∞}. This completes the proof. ▌

5. THE CASE OF A VARIABLE DAMPING WITH INDEFINITE SIGN

In this section we assume that there exist $x^-, x^+\in\Omega$ such that $A(x^-)<0$, $A(x^+)>0$ and that $\mathscr{L}$ satisfies the assumptions of Theorem 4.1. We shall show that if ε is sufficiently large, then $\mathscr{P}(\lambda)$ possesses at least four real zeros. Two negative and two positive. In particular, the zero solution of (1.1) is unstable. Then, we shall analyze the behavior of the eigenvalues and their associated eigenfunctions as ε goes to infinity. To prove these results we use Theorem 4.1.

THEOREM 5.1. *Suppose that there exists* $\beta \in C^2(\Omega)$ *such that* (4.1) *holds. Set*

$$\mathscr{P}_\varepsilon(\lambda) := \sigma_1^\Omega[\mathscr{L} + \varepsilon A(x)\,\lambda + \lambda^2]. \tag{5.1}$$

Then, for ε *large enough the function* $\mathscr{P}_\varepsilon(\lambda)$ *possesses at least two positive real zeros, say* $\lambda_{1,+}(\varepsilon) < \lambda_{2,+}(\varepsilon)$, *such that* $\lambda_{1,+}(\varepsilon) \downarrow 0$ *and* $\lambda_{2,+}(\varepsilon) \uparrow \infty$, *as* $\varepsilon \uparrow \infty$. *In fact,*

$$\lim_{\varepsilon \uparrow \infty} \varepsilon \lambda_{1,+}(\varepsilon) = \sigma_{1,+}^\Omega[\mathscr{L}; -A], \qquad \lim_{\varepsilon \uparrow \infty} \frac{\lambda_{2,+}(\varepsilon)}{\varepsilon} = -\inf_\Omega A, \tag{5.2}$$

where $\sigma_{1,+}^\Omega[\mathscr{L}; -A]$ *stands for the positve principal eigenvalue of the operator* $\mathscr{L}$ *with respect to the weight* $-A$. *Let* $\varphi_{j,+,\varepsilon}$ *denote the principal eigenfunctions associated with* $\lambda_{j,+}(\varepsilon)$, *i.e.,*

$$[\mathscr{L} + \varepsilon A(x)\,\lambda_{j,+}(\varepsilon) + \lambda_{j,+}^2(\varepsilon)]\,\varphi_{j,+,\varepsilon} = 0, \qquad j = 1, 2, \tag{5.3}$$

normalized so that

$$\int_\Omega |\nabla \varphi_{j,+,\varepsilon}|^2 = 1. \tag{5.4}$$

Then, for any sequence $\varepsilon_n \to \infty$, $n \to \infty$, *there exist two subsequences* (*relabeled by* n) *such that*

$$\lim_{n \to \infty} \varphi_{j,+,\varepsilon_n} = \varphi_{j,+,\infty}, \qquad j = 1, 2,$$

in $L^2(\Omega)$, *for some* $\varphi_{j,+,\infty} \in L^2(\Omega)$. *Moreover, the following holds*: (i) $\varphi_{2,+,\infty} \equiv 0$ *in the region where* $A > \inf_\Omega A$, (ii) $\varphi_{1,+,\infty}$ *is the principal eigenfunction associated with* $\sigma_{1,+}^\Omega[\mathscr{L}; -A]$. *Moreover,* $\lim_{\varepsilon \downarrow 0} \varphi_{1,+,\varepsilon} = \varphi_{1,+,\infty}$ *in* $L^2(\Omega)$.

Since A *changes of sign, the same result holds if we consider, instead of positive, negative eigenvalues of* $\mathscr{A}$. *Now, if we denote them by* $\lambda_{1,-}(\varepsilon) > \lambda_{2,-}(\varepsilon)$ *we have that*

$$\lim_{\varepsilon \uparrow \infty} \varepsilon \lambda_{1,-}(\varepsilon) = \sigma_{1,+}^\Omega[\mathscr{L}; A], \qquad \lim_{\varepsilon \uparrow \infty} \frac{\lambda_{2,-}(\varepsilon)}{\varepsilon} = -\sup_\Omega A. \tag{5.5}$$

The associated eigenfunctions satisfy analogous properties as above.

Remark 5.2. Since $\sigma_1^\Omega[\mathscr{L}]>0$ and A changes of sign, the equation $\mathscr{L}\varphi=\sigma A(x)\varphi$ subject to homogeneous Dirichlet boundary conditions possesses exactly two eigenvalues with a positive eigenfunction. One of them is positive, the other is negative. We have denoted them by $\sigma_{1,-}^\Omega[\mathscr{L};A]$ and $\sigma_{1,+}^\Omega[\mathscr{L};A]$ (c.f. [8], [12] and references therein). Observe that

$$-\sigma_{1,-}^\Omega[\mathscr{L};A]=\sigma_{1,+}^\Omega[\mathscr{L};-A], \qquad -\sigma_{1,+}^\Omega[\mathscr{L};A]=\sigma_{1,-}^\Omega[\mathscr{L};-A].$$

Proof of Theorem 4.2. We have that $\mathscr{P}_\varepsilon(0)=\sigma_1^\Omega[\mathscr{L}]>0$. Moreover,

$$\mathscr{P}_\varepsilon(\lambda)\geqslant\sigma_1^\Omega[\mathscr{L}]+\varepsilon\lambda\inf_\Omega A+\lambda^2\to\infty, \qquad \text{as} \quad \lambda\to\infty.$$

Thus, it suffices to show that if ε is sufficiently large, then $\mathscr{P}_\varepsilon(\lambda)<0$ for some $\lambda>0$. In such case we take

$$\lambda_{1,+}(\varepsilon)=\sup\{\lambda_1>0\colon \mathscr{P}_\varepsilon(\lambda)>0,\ \forall\lambda\in[0,\lambda_1)\},$$
$$\lambda_{2,+}(\varepsilon)=\inf\{\lambda_2>0\colon \mathscr{P}_\varepsilon(\lambda)>0,\ \forall\lambda\in(\lambda_2,\infty)\}.$$

We have that

$$\mathscr{P}_\varepsilon(\lambda)=\varepsilon\sigma_1^\Omega\left[\frac{1}{\varepsilon}\mathscr{L}+A(x)\,\lambda+\frac{\lambda^2}{\varepsilon}\right].$$

Moreover, thanks to Theorem 4.1,

$$\lim_{\varepsilon\uparrow\infty}\sigma_1^\Omega\left[\frac{1}{\varepsilon}\mathscr{L}+A(x)\,\lambda+\frac{\lambda^2}{\varepsilon}\right]=\lambda\inf_\Omega A<0.$$

From these features it follows easily that

$$\lim_{\varepsilon\uparrow\infty}\lambda_{1,+}(\varepsilon)=0, \qquad \lim_{\varepsilon\uparrow\infty}\lambda_{2,+}(\varepsilon)=\infty.$$

We now show (5.3). From the definition of $\lambda_{2,+}(\varepsilon)$ we have that

$$\sigma_1^\Omega\left[\frac{1}{\varepsilon\lambda_{2,+}(\varepsilon)}\mathscr{L}+A\right]=-\frac{\lambda_{2,+}(\varepsilon)}{\varepsilon}, \tag{5.6}$$

for ε is large enough. Since $\varepsilon\lambda_{2,+}(\varepsilon)\uparrow\infty$, as $\varepsilon\uparrow\infty$, it follows from Theorem 4.1 that the limit of the left hand side of (5.6) is $\inf_\Omega A$. This completes the proof of the second relation of (5.2). We now complete the proof of (5.2). By definition,

$$\sigma_1^\Omega[\mathscr{L}+\varepsilon\lambda_{1,+}(\varepsilon)A]=-\lambda_{1,+}^2(\varepsilon). \tag{5.7}$$

We claim that the family $(\varepsilon\lambda_{1,+}(\varepsilon))$ is bounded. We shall argue by contradiction. Assume that $\lim_{n\to\infty}\varepsilon_n\lambda_{1,+}(\varepsilon_n)=\infty$ for some sequence ε_n such that $\lim_{n\to\infty}\varepsilon_n=\infty$. Then, since A is negative in a subdomain of Ω we have that

$$\lim_{n\to\infty}\sigma_1^{\Omega}[\mathscr{L}+\varepsilon_n\lambda_{1,+}(\varepsilon_n)A]=-\infty.$$

On the other hand, by definition we have that $\lim_{n\to\infty}\lambda_{1,+}(\varepsilon_n)=0$ and this is impossible, because of (5.7). Thus, $(\varepsilon\lambda_{1,+}(\varepsilon))$ is bounded. From any sequence of this family we can extract a convergent subsequence to some $L\geqslant 0$. Restricting (5.7) along such subsequence and passing to the limit we find that

$$\sigma_1^{\Omega}[\mathscr{L}+LA]=0.$$

Therefore, $L=\sigma_{1,+}^{\Omega}[\mathscr{L};-A]$. This completes the proof of (5.2).

We now analyze the behavior of the associated eigenfunctions, $\varphi_{1,+,\varepsilon}$ and $\varphi_{2,+,\varepsilon}$. It follows from the definition of $\varphi_{2,+,\varepsilon}$ that it is the principal eigenfunction associated with $\sigma_1^{\Omega}[1/(\varepsilon\lambda_{2,+}(\varepsilon))\,\mathscr{L}+A]$. Moreover, $\varepsilon\lambda_{2,+}(\varepsilon)\to\infty$ as $\varepsilon\to\infty$. Therefore, all the assertions of the theorem follow from Theorem 4.1. To prove the assertions concerning with $\varphi_{1,+,\varepsilon}$ we argue as follows. The normalization condition (5.4) together with Poincaré inequality imply that $(\varphi_{1,+,\varepsilon})$ is bounded in $H_0^1(\Omega)$. Therefore, from any sequence of this family we can substract a subsequence convergent in L^2 to some $\varphi_{1,+,\infty}\in L^2$. Let $\varphi_{1,+,\varepsilon_n}$ denote such a subsequence. We now show that $(\varphi_{1,+,\varepsilon_n})$ is Cauchy in $H_0^1(\Omega)$. To simplify the notation we denote

$$\phi_n:=\varphi_{1,+,\varepsilon_n},\qquad p_n:=-\lambda_{1,+}^2(\varepsilon_n)-\varepsilon_n\lambda_{1,+}(\varepsilon_n)\,A-\alpha_0,\qquad n\geqslant 1. \quad (5.8)$$

Note that $\lim_{n\to\infty}\lambda_{1,+}^2(\varepsilon_n)=0$ and $\lim_{n\to\infty}\varepsilon_n\lambda_{1,+}^2(\varepsilon_n)=\sigma_{1,+}^{\Omega}[\mathscr{L};-A]$. So,

$$\lim_{n\to\infty}p_n=-\sigma_{1,+}^{\Omega}[\mathscr{L};-A]\,A-\alpha_0$$

in $L^{\infty}(\Omega)$. In particular, (p_n) is bounded in $L^{\infty}(\Omega)$. By definition,

$$-\sum_{i,j=1}^{N}D_j(\alpha_{ij}\,D_i\phi_n)+\sum_{j=1}^{N}\alpha_j\,D_j\phi_n=p_n\phi_n,\qquad n\geqslant 1. \quad (5.9)$$

Using (2.6), integrating by parts and rearranging terms, it follows from (5.9) that for any $k,l\geqslant 1$

$$
\begin{aligned}
\nu \int_\Omega |\nabla(\phi_k-\phi_l)|^2 &\leqslant \sum_{i,j=1}^N \int_\Omega \alpha_{ij}\, D_i(\phi_k-\phi_l)\, D_j(\phi_k-\phi_l) \\
&= \sum_{i,j=1}^N \int_\Omega \alpha_{ij}\, D_i\phi_k\, D_j\phi_k + \sum_{i,j=1}^N \int_\Omega \alpha_{ij}\, D_i\phi_l\, D_j\phi_l \\
&\quad -2 \sum_{i,j=1}^N \int_\Omega \alpha_{ij}\, D_i\phi_k\, D_j\phi_l \\
&= \int_\Omega \left(p_k\phi_k - \sum_{j=1}^N \alpha_j\, D_j\phi_k \right) \phi_k + \int_\Omega \left(p_l\phi_l - \sum_{j=1}^N \alpha_j\, D_j\phi_l \right) \phi_l \\
&\quad -2 \int_\Omega \left(p_k\phi_k - \sum_{j=1}^N \alpha_j\, D_j\phi_k \right) \phi_l.
\end{aligned}
$$

Rearranging terms gives

$$
\begin{aligned}
\nu \int_\Omega |\nabla(\phi_k-\phi_l)|^2 \leqslant \int_\Omega p_k\phi_k(\phi_k-\phi_l) + \int_\Omega p_l\phi_l(\phi_l-\phi_k) \\
+ \int_\Omega (p_l-p_k)\, \phi_l\phi_k + \sum_{j=1}^N \int_\Omega \alpha_j(\phi_l-\phi_k)\, D_j\phi_k \\
+ \sum_{j=1}^N \int_\Omega \alpha_j\phi_l\, D_j(\phi_k-\phi_l). \qquad (5.10)
\end{aligned}
$$

In the sequel we shall denote by C_1, C_2, C_3, ... to some constants which do not vary with k and l, whose explicit knowledge is not important for our purposes here. From Hölder inequality, it can be easily seen that for any $k, l \geqslant 1$,

$$
\left| \int_\Omega p_k\phi_k(\phi_k-\phi_l) \right| \leqslant \|p_k\phi_k\|_{L^2(\Omega)}\, \|\phi_k-\phi_l\|_{L^2(\Omega)} \leqslant C_1\, \|\phi_k-\phi_l\|_{L^2(\Omega)},
$$

$$
\left| \int_\Omega p_l\phi_l(\phi_l-\phi_k) \right| \leqslant C_2\, \|\phi_k-\phi_l\|_{L^2(\Omega)},
$$

$$
\left| \int_\Omega (p_l-p_k)\, \phi_l\phi_k \right| \leqslant \|p_l-p_k\|_{L^\infty}\, \|\phi_k\|_{L^2}\, \|\phi_l\|_{L^2} \leqslant C_3\, \|p_l-p_k\|_{L^\infty},
$$

$$
\left| \sum_{j=1}^N \int_\Omega \alpha_j(\phi_l-\phi_k)\, D_j\phi_k \right| = \left| \int_\Omega (\phi_l-\phi_k)\langle \vec{\alpha}, \nabla\phi_k \rangle \right| \leqslant C_4\, \|\phi_k-\phi_l\|_{L^2(\Omega)},
$$

$$
\left| \sum_{j=1}^N \int_\Omega \alpha_j\phi_l\, D_j(\phi_k-\phi_l) \right| = \left| \sum_{j=1}^N \int_\Omega (\phi_k-\phi_l)\, D_j(\alpha_j\phi_l) \right| \leqslant C_5\, \|\phi_k-\phi_l\|_{L^2(\Omega)}.
$$

Thus, we find from (5.10) that (ϕ_n) is Cauchy in $H_0^1(\Omega)$. Therefore, its limit $\phi_\infty := \varphi_{1,+,\infty} \in H_0^1(\Omega)$. Let $\psi \in \mathscr{D}(\Omega) := C_0^\infty(\Omega)$ arbitrary. Multiplying (5.9) by ψ, integrating over Ω and applying the formula of integration by parts we find that

$$\sum_{i,j=1}^{N} \int_\Omega \alpha_{ij} D_i\phi_n D_j\psi + \sum_{j=1}^{N} \int_\Omega \alpha_j \psi D_j\phi_n = \int_\Omega \psi p_n \phi_n, \qquad n \geqslant 1.$$

Thus, passing to the limit as $n \to \infty$ we obtain that ϕ_∞ is a weak solution of

$$\mathscr{L}\phi_\infty + \sigma_{1,+}^\Omega [\mathscr{L}; -A] A\phi_\infty = 0,$$

and therefore, ϕ_∞ is the principal eigenfunction associated with $\sigma_{1,+}^\Omega[\mathscr{L}; -A]$. Note that this is true along any subsequence of $(\varphi_{1,+,\varepsilon})$. The simplicity of $\sigma_{1,+}^\Omega[\mathscr{L}; -A]$ completes the proof, [8]. ∎

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